Reflexive dg categories & $HH^\bullet$

loosely based on joint work with Isambard Goodbody & Sebastian Opper.

Def $\mathcal{A}$ dg category $/ K$

$\mathcal{A}\text{-Mod} = \mathrm{dgFun}(\mathcal{A}, \mathrm{Ch}(K))$

$D(\mathcal{A}) = \mathcal{A}\text{-Mod}[\text{quasi-isos}^{-1}]$

Say that an $\mathcal{A}$-module M is <u>proper</u> if $\forall a \in \mathcal{A}$, $M(a) \in \mathrm{per}(K)$

These assemble into a category

$D_{fd}(\mathcal{A}) \subseteq D(\mathcal{A})$

Thm (Ballard '11)

X proj. variety $/ K$

Then D_{fd} permutes $\mathrm{per}X$ & $D^b_{coh}X$

Th^m (Ben-Zvi – Nadler – Preygel '17)
Same result for (relative) alg. space (in char 0)

Th^m (Chen '21)
Same is true for fin. dim. algs.

Expectation In all of these examples, $HH^\bullet(C) \simeq HH^\bullet(D_{fd}C)$
(Keller, Lowen – Van den Bergh, Toën, ...)

Def (Kuznetsov – Shinder '22)
First observe that there's a natural evaluation map

$$ev : A \longrightarrow D_{fd} D_{fd} A$$
$$a \longmapsto (M \mapsto M(a))$$

Say that A is <u>reflexive</u> if ev is a derived Morita equivalence

$\iff \operatorname{per} A \longrightarrow D_{fd} D_{fd} A$ being a quasi-equivalence.

All of the previous examples are reflexive!

<u>Rmk</u> We have a quasi-equivalence

$$D_{fd}(A) \simeq \mathbb{R}\mathrm{Hom}_{Hqe}(A, \operatorname{per} k) \simeq \mathbb{R}\mathrm{Hom}_{Hmo}(A, k)$$

<u>Thm</u> (Goodbody '24)

C reflexive $\Rightarrow HH^\bullet C \simeq HH^\bullet D_{fd} C$

$\Rightarrow \operatorname{DPic} C \simeq \operatorname{DPic} D_{fd} C$

$\Rightarrow HH_\bullet C \simeq HH_\bullet D_{fd} C$

$K[t]$

Other known examples of reflexive dg cats

- smooth proper dg cats
  (since $\mathrm{per} A \simeq D_{fd} A$)
- Proper connective dg algs
- Proper schemes / any field
- $K[[t]]$ but not $K[t]$
- Fukaya cats of Milnor fibres of certain weighted hom. polys.

in [BGO]:

i) characterisation of reflexivity for dg algebras A in terms of

- generators for $D_{fd}(A)$
- Being derived complete
  with respect to these generators.
  (dc-complete ⇝ Koszul duality).

ii) Apply this to give new examples

- char. when Noetherian comm. rings are reflexive
- dg algebras associated to silting objects / simple-minded collections are reflexive
- Completed Ginzburg dg algs are reflexive
- Fukaya cats of surfaces / gentle algebras are reflexive
- Algs associated to top. spaces

Topological examples

X top space (path connected)

$C_\bullet(X, K) \rightsquigarrow$ chains on X

$C^\bullet(X, K) \rightsquigarrow$ cochains on X

$C_\bullet(\Omega X, K) \rightsquigarrow$ chains on loops.

$$D_{fd}(C_\bullet \Omega X) \simeq \left\{ \begin{array}{l} \infty\text{-local systems of} \\ \text{vector spaces on } X \\ \text{with finite fibres} \end{array} \right\}$$

$$D_{fd}\, C^\bullet X \simeq \mathrm{per}\left((C_\bullet \Omega X)^{!!} \right)$$

double Koszul dual / derived completion

Prop (BGO)

X either • finite simply connected CW complex

or • BG, G finite p-group (K char p)

Then $C^\bullet X$ & $C_\bullet \Omega X$ are reflexive & therefore each others Morita duals

(e.g. $C_\bullet \Omega X \simeq_{Mor} D_{fd}\, C^\bullet X$)

Cor: In the above setting, every proper $C^\bullet X$ - module dualises to a $C_\bullet X$ - comodule

String topology flavoured example

M compact orientable simply conn. manifold.

Jones: $HH_\bullet C_\bullet \Omega M \simeq C_\bullet LM$

can show

$HH^\bullet(D_{fd}\, C^\bullet M) \simeq HH^\bullet(D_{fd}\, C_\bullet \Omega M)$

$$\simeq C_\bullet(LM)[-\dim M]$$

(Tradler - Zeinalian)