Associative Koszul Duality

Yonsei University, June '26

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§1 MOTIVATION

Def^n thy in AG - infinitesimal def^{n}s

Infinitesimals are $\mathrm{cArt}_k \ni k[\varepsilon]/\varepsilon^2$

A def of X over Spec Λ is flat $\mathfrak{X} \to \operatorname{Spec}\Lambda$ st $\mathfrak{X} \otimes_\Lambda k \simeq X$

Ex) $\dfrac{k[x,y,\varepsilon]}{xy-\lambda\varepsilon}$ is a 1st order def of $\dfrac{k[x,y]}{xy}$.

Abstracted into a deformation functor

$\mathrm{cArt}_k \xrightarrow{F} \mathrm{Set}$ satisfying $F(k) = *$

& some descent/gluing properties

Better: use derived def. functors

aka formal moduli problems

i.e. functors $\mathrm{cdgArt}_k^{\leq 0} \longrightarrow \mathrm{sSet}$

w/ some descent properties.

(Better since e.g. they sheafify)

Thm (Lurie-Pridham correspondence) ← after Deligne

$\exists$ an equivalence of ∞-cats

$$\mathrm{FMP} \longrightarrow \mathrm{dgLie}$$

Proof idea Calculus!

a FMP is some sort of formal stack so has a tangent complex T which is a (shifted) dgla.

Can integrate T formally locally.

In fact, this is more or less the Koszul duality between Com & Lie.

Th^m (noncommutative version - Lurie)

$$\mathrm{ncFMP} \xleftrightarrow{\simeq} \mathrm{dgAlg}^{\mathrm{aug}}$$

$$\cap\!\!|$$

$$\mathrm{Fun}(\mathrm{dgArt}_k^{\mathrm{co}}, \mathrm{sSet})$$

this is the associative-associative KD

why coalgebras? Useful to obtain various representability results for FMPs. Idea: dgCog is best approx. to dgAlg using just dgArt_k.

§2. DG ALGEBRAS & COALGEBRAS

Recall a dg Alg is a monoid in $(Ch(k), \otimes)$, ie A with $k \to A$ & $A \otimes A \to A$

A is augmented if $\exists$ $A \to k$ splitting the unit.

μ is a chain map $\Leftrightarrow$ Leibniz rule

Def a dgCog is a comonoid in $Ch(k)$ ie C with $C \xrightarrow{\eta} k$ & $C \xrightarrow{\Delta} C \otimes C$ + coAss + coUn

Δ is a chain map $\Leftrightarrow$ d is a coderivation

$$\Delta d = (d \otimes id + id \otimes d)\Delta$$

a morphism of coalgebras is a morphism of comonoids.

Lem if C is a dgc then C^* is a dga.

If A is a fin dim dga then A^* is a dga

$$dgCog_{fd} \simeq dgAlg_{fd}^{op} \text{ via } *.$$

Proof

$$V^* \otimes W^* \to (V \otimes W)^*$$
$$f \otimes g \mapsto [v \otimes w \mapsto f(v)g(w)]$$

A dg Cog is coaugmented if $C \to k$ has a coalgebra split.

In this case, $\bar{C} = \ker(C \to k) \simeq \operatorname{coker}(k \to C)$ is a noncounital dg Cog w/ reduced comultiplication $\bar{\Delta} : \bar{C} \to \bar{C} \otimes \bar{C}$

Def a coaug C is conilpotent if $\forall c \in \bar{C}$ $\exists N$ $\bar{\Delta}^N c = 0$.

Def V a dg vs. the **tensor coalgebra** of V is $T^cV = k \oplus V \oplus V^2 \oplus \cdots$

w/ **deconcatenation coproduct**

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_i v_1 \otimes \cdots v_i \otimes v_{i+1} \otimes \cdots \otimes v_n$$

It's conilpotent.

In fact T^cV is the **cofree conil coalg** on V in the sense that

$$\mathrm{Cog}^{\mathrm{conil}}(C, T^cV) \simeq \mathrm{Vect}(C, V)$$

Rmk **cofree coalgebra** exist but are nasty.

§2½ PSEUDOCOMPACT ALGEBRAS

Def $\mathcal{C}$ a cat.

$\operatorname{ind}\mathcal{C}$ = "formal filt. colims in $\mathcal{C}$"

ie objects = $I \to \mathcal{C}$ w/ I filtered or directed

$$\operatorname{ind}\mathcal{C}(X,Y) = \varprojlim_i \varinjlim_j \operatorname{Hom}(X_i, Y_j)$$

Similarly $\operatorname{pro}\mathcal{C} = \operatorname{ind}(\mathcal{C}^{op})^{op}$

"formal cofilt. lims in $\mathcal{C}$" or inverse

$$\operatorname{pro}\mathcal{C}(X,Y) = \varprojlim_j \varinjlim_i \operatorname{Hom}(X_i, Y_j)$$

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if $\mathcal{C}$ has filt colims, have $\varinjlim : \operatorname{ind}\mathcal{C} \to \mathcal{C}$

Always have $\mathcal{C} \to \operatorname{ind}\mathcal{C}$

$$c \longmapsto [* \xrightarrow{c} \mathcal{C}]$$

Ex Every vs is a colim of its fd subspaces

So $\varinjlim : \mathrm{ind}\,\mathrm{Vect}_{fd} \longrightarrow \mathrm{Vect}$ is ess. surj.

It's also fully faithful since every object of Vect_{fd} is compact.

So $\mathrm{ind}\,\mathrm{Vect}_{fd} \xrightarrow{\sim} \mathrm{Vect}$

& $\mathrm{pro}(\mathrm{Vect}_{fd}^{op}) \xrightarrow{\simeq} \mathrm{Vect}^{op}$

But $\mathrm{Vect}_{fd}^{op} \simeq \mathrm{Vect}_{fd}$ via $*$

$\boxed{\text{So}}$ $\mathrm{Vect}^{op} \simeq \mathrm{pro}\,\mathrm{Vect}_{fd}$ "profinite vector spaces"

Thm $\mathrm{dgCog} \simeq \mathrm{ind}(\mathrm{dgCog}_{fd})$

& $\mathrm{dgCog}^{op} \simeq \mathrm{pro}(\mathrm{dgAlg}_{fd})$

"pseudocompact dg algebras"

Also for coaug/aug conil/nil

Proof idea 1) Fundamental Thm of coalgebras

: every $x \in C$ is contained in a fin dim $C' \subseteq C$ subcoalg [Proof: some easy algebra]

2) fin dim coalgs are compact

As before we conclude $\mathrm{dgCog} = \mathrm{ind}\,\mathrm{dgCog}_{fin}$

& since $\mathrm{dgCog}_{fd}^{op} \simeq \mathrm{dgAlg}_{fd}^{op}$

we conclude $\mathrm{dgCog}^{op} \simeq \mathrm{pro}\,\mathrm{dgAlg}_{fd}$. $\square$

Rmk $\mathrm{pcAlg} := \mathrm{pro}\,\mathrm{dgAlg}_{fd}$

Inverse limits set discrete topology

$\Longrightarrow$ $\mathrm{pcAlg} \xhookrightarrow{\text{ff.}} \mathrm{TopAlg}$

Ex a fin dim alg is pc

$K[[x]]$ with its usual top is pc

Dual is $T^c(x^*) \simeq_{vect} K[x^*]$

More generally, $T(V)^* = K\langle\langle V^* \rangle\rangle$

when V is inf. dim, have to interpret appropriately as

$$\widehat{K \oplus V^* \oplus (V^* \hat{\otimes} V^*) \oplus \cdots}$$

§3 BAR & COBAR

Def A an augmented dg alg.

$$BA = \left(T^c(\bar{A}[1]),\ d_A + \partial\right)$$

∂ is the Bar differential

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_i (-1)^i a_1 \otimes \cdots a_i a_{i+1} \otimes \cdots \otimes a_n$$

(Lemma: $d+\partial$ is a differential) Pf $\partial^2 = 0$ since μ is assoc

Clearly BA is conilpotent.

Def Similarly, C a conil. dg Cog.

$$\Omega C = \left(T(\bar{C}[-1]),\ d_c + \partial\right)$$

$$\partial(c_1 \cdots c_n) = \sum_i (-1)^i c_1 \cdots \Delta c_i \cdots c_n$$

Ex. Put $A^! = (BA)^*$ dg Alg

In fact $A^! \simeq R\mathrm{End}_A(k)$

(bar resolution) or mod-comod KD

So $\mathrm{thick}_{DA}(k) \simeq \mathrm{per}\, A^!$

This is a weak version of module-comodule KD.

Ex] X a pc top. space

$C_\cdot X = C_\cdot(X, K)$ dgCog $(\Delta : X \to X \times X)$

$C_\cdot \Omega X = C_\cdot(\Omega X, K)$ dgAlg (Moore loops)

Adams '59 ($\pi_1 = 0$), Rivera-Zeinalian '17

$$\Omega C_\cdot X \simeq C_\cdot \Omega X$$

this is why cobar is denoted with Ω.

Thm $\Omega \dashv B$

Proof requires some new notions.

Def C dgc, A Aga

the convolution algebra is

$\mathrm{Hom}(C, A)$ with multiplication

$$f \cdot g : \quad C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

Def E a dgAlg. a Maurer-Cartan elt

in E is $x \in E^1$ st $dx + x^2 = 0$

get a set $MC(E) \subseteq E^1$

Rmk x MC $\Longleftrightarrow$ $d_x = d + [-, x]$ is a diff

Rmk in char $\neq 2$, $x^2 = \frac{1}{2}[x, x]$

so MC eqn is $dx + \frac{1}{2}[x, x] = 0$.

Proof of adjunction

"twisting cochains"

We show $\mathrm{Hom}(\Omega C, A) \simeq \mathrm{MCHom}(\bar{C}, \bar{A})$

$\simeq \mathrm{Hom}(C, BA)$

Only show first iso; second is similar.

Forgetting the differential,

$\mathrm{grAlg}^{aug}(\Omega C, A) \simeq \mathrm{Hom}^1(\bar{C}, \bar{A})$ since $(\Omega C)^{\#}$ is free on $\bar{C}[-1]$.

Compatibility with differential $\Leftrightarrow$ MC elements.

Thm A a dgAlg^{aug}:

$\Omega BA \longrightarrow A$ is a cofibrant dgAlg resolution.

Proof Just need to show is a $\simeq$iso

Idea: spectral sequence argument + explicit contracting htpy. / Alt: follows from mod-comod KD

Rmk) B preserves q isos (spec. seq ...)

Ω does not (see $C_\bullet S^1$ example)

But say $C \to C'$ is a **weak equivalence** if $\Omega C \to \Omega C'$ is a q iso.

then) B sends q isos to WEs

$$\begin{tikzcd} A \arrow[r] & A' \\ \Omega B A \arrow[r] \arrow[u] & \Omega B A' \arrow[u] \end{tikzcd}$$

Ω sends WEs to q isos clear.

Thm $C \to B\Omega C$ is a weak equiv

Proof Zigzag identities for adjunctions

$\Longrightarrow$ $\Omega C \to \Omega B \Omega C$ has a retract given by $\Omega B(\Omega C) \to \Omega C$ which we know is a q iso.

Ex X a pc Top space

$$(C_\cdot \Omega X)' \simeq (BC_\cdot \Omega X)^*$$
$$\simeq (B\Omega C_\cdot X)^*$$
$$\simeq (C_\cdot X)^*$$
$$= C^\cdot X$$

Ex $C = C_\cdot S'$

$$D = \left(\frac{K[\varepsilon']}{\varepsilon^2}\right)^*$$

Then $C \simeq_{qi} D$ but $C \not\simeq_{we} D$

since $\Omega C \simeq C_\cdot \Omega S' \simeq C_\cdot \mathbb{Z} \simeq K[t, t^{-1}]$

but $\Omega D \simeq K[t]$

Def) a twisting cochain $\tau \in MCHom(C,A)$ is **acyclic** if the corr. map $\Omega C \to A$ is a qiso

Lem τ acyclic $\iff$ $C \to BA$ is a WE

Proof τ acy $\Rightarrow$ $B\Omega C \to BA$ WE

But $C \to BA$ factors $\rightsquigarrow$ $C \xrightarrow{WE} B\Omega C \xrightarrow{WE} BA$

$C \to BA$ WE $\Rightarrow$ $\Omega C \to \Omega BA$ qiso

But $\Omega C \xrightarrow{\simeq} \Omega BA \xrightarrow{\simeq} A$

Thm (Positselski) (cf Quillen, Hinich, Lefèvre-Hasegawa, ...)

$dgCog^{conil}$ is a model category & bar-cobar is a Quillen equivalence

$$\Omega : dgCog^{conil} \longleftrightarrow dgAlg^{aug} : B$$

usual mo. str.

In particular, we get an equivalence of ∞-cat^s

$$\Omega : dgCog^{conil}[WE^{-1}] \longleftrightarrow dgAlg^{aug}[qiso^{-1}] : B$$

§4 MODULE-COMODULE KD

after Positselski.

Def C a dg Cog

a **dg-C-comodule** is a dg vs N

with a **coaction map** $\rho : N \to N \otimes C$

s.t.

$$\begin{tikzcd} N \arrow[r, "\rho"] \arrow[d, "\rho"'] & N \otimes C \arrow[d, "id \otimes \Delta"] \\ N \otimes C \arrow[r, "\rho \otimes id"] & N \otimes C \otimes C \end{tikzcd} \quad \& \quad \begin{tikzcd} N \arrow[r, "\rho"] \arrow[dr, "id"'] & N \otimes C \arrow[d, "id \otimes \eta"] \\ & N \end{tikzcd}$$

Morphisms are linear maps respecting coactions

Rmk $*: C\text{-Comod}^{op} \to C^*\text{-Mod}$

far from being an equivalence

Rmk $C\text{-Comod} = \operatorname{ind}(C\text{-Comod}_{fd})$

via a similar fundamental thm.

C-Comod is naturally a dg category & so has a homotopy category $H^0(C\text{-Comod})$ which is triangulated

(analogue: $K(A\text{-Mod})$)

closed under $\oplus_s$

Def The smallest localising subcat of $H^0(C\text{-Comod})$ containing totalisations of exact triples of comodules is the subcat of <u>coacyclic</u> comodules $Acy^{co}C$

Def
$$D^{co}C := \frac{H^0(C\text{-Comod})}{Acy^{co}C}$$
is the **coderived category** of C

Rmk Linear dual gives a functor

$$D^{co}C \xrightarrow[\text{loc}^n]{} D(C\text{-Comod}) \to D(C^*)^{op}$$

First is far from being fully faithful since it kills all coacyclic comods

Second fails to be both ff & ess surj.

In fact $Acy^{co}C = {}^{\perp}C$

Under some set-theoretic assumptions (Vopěnka's principle) we have

$$\mathrm{Coloc}_{D^{co}C}(C) \simeq D(\mathrm{Comod}\text{-}C)$$

& a SOD

$$D^{co}C = \langle D(C\text{-Comod}), Acy^{co}C \rangle$$

Injective resolutions

a C-comod N is <u>injective</u> if $N^\#$ is an injective graded $C^\#$-comodule (summand of a cofree $C \otimes V$)

Prop

C-Inj $\hookrightarrow$ C-Comod induces an equivalence

$$H^0(C\text{-Inj}) \to H^0(C\text{-Comod}) \to D^{co}(C)$$

Proof idea

1) every comod N has a cobar rsl^n $C \otimes N \to C \otimes C \otimes N \to \cdots$ which is injective.

2) $\mathrm{Hom}(\mathrm{Acy}^{co} C, C\text{-Inj}) \simeq 0$.

Prop $C\text{-Comod}_{fd}$ is a set of compact generators for $D^{co}C$.

Proof want to show

1) fin dim comods are cpct
2) $D^{co}C = \mathrm{Loc}(C\text{-Comod}_{fd})$

1) uses description of $D^{co}C = H^0(C\text{-Inj})$

2) Idea: any comodule is a filt. colim. of fin. dim comods.

Rmk C has a <u>coradical filtration</u>

showing that C is conilpotent over some cosemisimple dg coalgebra.

This passes to comodules, & in fact
$C\text{-Comod}_{fd} = \mathrm{thick}(C_{css})$

So when C is conilpotent ($C_{css} = k$) we see that the C-comodule k is a cpct generator of $D^{co}C$.

Intuition: coalgebras & their comod's behave like fin dim algs & fin dim mods! e.g. if A is a fin dim alg then

$$D^b(A\text{-mod}) = \text{thick}(A/\text{rad}A)$$

via the radical filtration.

Sweedler notation

Write $\Delta c = c^1 \otimes c^2$ as shorthand for

$$\Delta c = \sum_i c^1_i \otimes c^2_i.$$

Similarly if $\rho: N \to C \otimes N$ a comod write $\rho n = n^1 \otimes n^2$ with $n^1 \in C$, $n^2 \in N$

Twists

From now on, fix a triple (C, A, τ) with $\tau : C \to A$ an acyclic twisting cochain (ie $\Omega C \xrightarrow{\sim} A$ & $C \xrightarrow{\sim} BA$)

If M is an A-module, get a C-comodule $M \otimes_\tau C$ with differential twisted by the term

$$m \otimes c \longmapsto \pm m \cdot \tau(c^1) \otimes c^2$$

Similarly if N is a C-comodule, get an A-module $N \otimes_\tau A$ w/ diff. twisted by

$$n \otimes a \longmapsto \pm n^1 \otimes \tau(n^2) a$$

Th^m (Positselski)

The functors $M \mapsto M \otimes_\tau C$

$N \mapsto N \otimes_\tau A$

give mutually inverse triangle functors

$$D^{co}(C) \simeq D(A)$$

in fact this comes from a Quillen equiv. of model cat^s.

Proof 1) If N is coacyclic then $N \otimes_\tau A$ is contractible [check on exact triples]

2) if M is acyclic then $M \otimes_\tau C$ is coacyclic [filter by tensor length on C]

$1+2 \Rightarrow$ have functors $D^{co}C \rightleftarrows DA$

3) $- \otimes_\tau A \dashv - \otimes_\tau C$ [not too hard]

Need to check

4) $M \otimes_\tau C \otimes_\tau A \simeq_{qi} M$

[LHS is bar rslⁿ of M]

5) $N \otimes_\tau A \otimes_\tau C \simeq_{we} N$

[another filtration argument]

Rmk

can check $D^{co}C \overset{\simeq}{\longleftrightarrow} DA$

$$K \longleftrightarrow A$$

$$C \longleftrightarrow K$$

using bar rslⁿs.

In fact it's an equiv. of dg cat^s so we get

$A \simeq \mathbb{R}End_{D^{co}C}(K)$

$C^* \simeq \mathbb{R}End_{DA}(K)$

§4½ CONTRAMODULES

A **contramodule** $/C$ is a dgvs P with an appropriately associative map

$$\mathrm{Hom}(C,P) \longrightarrow P$$

Ex if N is a C-comod then $\mathrm{Hom}(N,V)$ is a C-ctrmod for any dgvs V

The **free** ctrmods are ~~those~~ of the form $\mathrm{Hom}(C,V)$

Dually as before, these fit into a **contraderived category**

$$D^{ctr}C = \frac{H^0(C\text{-}ctr)}{Acy^{ctr}C}$$

colocalising on totalisations of SESs

& for any C at all we have

$$D^{ctr} C \simeq D^{co} C$$

$$N \longmapsto \mathrm{Hom}_C(C, N)$$

$$C \odot P \longleftarrow P$$

this is Positselski's

comodule - contramodule correspondence

Koszul triality:

an explicit twist of the form $\mathrm{Hom}^{\tau}(C, -)$

$$DA \simeq D^{co} C \simeq D^{ctr} C$$

$$K \longleftrightarrow C \longleftrightarrow C^*$$

$$A \longleftrightarrow K \longleftrightarrow K$$

$$A^* \longleftrightarrow C$$

§5 Possible extras

1) Positselski has KD for

$$\mathrm{cuCog}^{\mathrm{conil}} \longleftrightarrow \mathrm{dgAlg}$$

both module-comodule & algebra-coalgebra

2) More on def^m theory (DAGX ...)